Dimensions and Economics: Some Answers

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William Barnett's (2004) critique of mathematics in economic analysis, "Dimensions and Economics: Some Problems," claims that economics almost always uses functions and equations without paying any attention to their variable and parameter dimensions and units. By casual observation, that criticism appears often to be true,[1] and it applies not only to functions and equations but also to relations of all sorts, including inequalities.

Following his introduction, Barnett (p. 95) makes his main case in three sections, using two different examples: The Cobb-Douglas production function, whose parameters he sees as having (1) "meaningless or economically unreasonable dimensions" (p. 96) and (2) "inconstant" dimensions (p. 97). A macroeconomic model, whose technology (production) function's parameters

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[1]This inattention to dimensional analysis begins with introductory economics (and business) textbooks, in which function examples usually ignore dimensions. For example, textbooks routinely include demand and supply functions such as $q^d = 10 - 3p$ or $q^s = 2 + 5p$, with no mention that 10, -3, 2, and 5 all are parameter values that must have dimensions, such that the function's left side dimension is matched by the function's right side dimension. In this case, the dependent variable has quantity dimension "quantity unit" (e.g., bushels), and the independent variable has price dimension "monetary unit / quantity unit" (e.g., dollars/bushel). Hence the freestanding additive parameters 10 or 2 each must have the dimension "quantity unit," and the (logical) slope parameters -3 or 5 each must have the dimension "(quantity unit)^2 / monetary unit." Far better is to introduce functions in a more general form, such as $q^d = D(p) = d_0 + d_1 p$ (where $d_1 < 0$) and $q^s = S(p) = s_0 + s_1 p$, and then discuss parameter dimensions and suggest reasonable parameter values.

he sees as either (3) "meaningless," or else as defining a function that relates output hours to labor input hours, which yields "no net production." Either way, the model is "not defensible" (pp. 97–98).

Professor Barnett's fourth and fifth sections, "Discussion" (pp. 98–99) and "Conclusions" (p. 99), state that

> [Modern economics'] failure to use dimensions consistently and correctly in production [and other economic] functions . . . are both critical and ubiquitous—they afflict virtually all mathematical and econometric models of economic activity. . . . [T]he failure to use dimensions consistently and correctly . . . render the models so afflicted virtually worthless. . . .
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> [Economics has been "emulating" the methods of "physicists and engineers," but has "failed to emulate" them in] the consistent and correct use of dimensions. This is an abuse of mathematical/scientific methods. Such abuse invalidates the results of mathematical and statistical methods applied to the development and application of economic theory. . . . [I]t is a continuing problem and one found in the leading mainstream journals (and textbooks). . . . [U]nless and until this changes, and economists consistently and correctly use dimensions in economics, if such is possible, mathematical economics, and its empirical alter ego, econometrics, will continue to be academic games and "rigorous" pseudosciences. . . .
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> This is not to say that there have not been advances in economic understanding by the neoclassicals, but rather to argue that mathematics is neither a necessary nor a sufficient means to such advances. Whether it even is, or can be, a valid means to such advances is a different issue.

Professor Barnett makes some good points in his critique of the use of mathematics in economics. Criticism of mathematical economics, however, is hardly a novel topic among Austrian economists (Mises 1977; 1966, pp. 350–57).[2] And the misuse and abuse of mathematics in contemporary economics has been noticed, denounced, and lamented even by prominent mainstream economists (see Blaug 1998, pp. 11–34, and his citations). What is novel in Barnett's article is his claim that *dimensional* errors in the mathematical functions used in economics are "ubiquitous," which makes a lot of mainstream economic models "worthless." Although much and perhaps most contemporary neoclassical economics may be worthless, that assessment has long preceded Barnett's dimensional issue.

In our judgment, Barnett's discussion of the Cobb-Douglas production function and of the macroeconomic model merits further examination. That economics often ignores dimensions and units does not necessarily mean that unstated but implied dimensions and units are wrong or invalid.

For evidence that economists (including Maurice Allais, Hans Brems, Nicholas Georgescu-Roegen, and less completely, Gardner Ackley and Kenneth Boulding) have paid at least *some* attention to dimensional analysis, see De Jong (1967, pp. 1–3), and his references to help from other economists (1967, ix–x).

[2]See also Leoni and Frola (1977, including their note 3, p. 109, which has quotations and references from Henry Hazlitt [1959]).

For each of Professor Barnett's three sections discussing his specific examples, our comments are in sections numbered 1, 2, and 3, respectively. We conclude with some "Final Thoughts," about the use of mathematics in economics.[3]

1. Cobb-Douglas Parameter *Dimensions* that are not Meaningless, Although Some Parameter *Values* Are Unreasonable

Professor Barnett writes the two-input Cobb-Douglas production function in the usual notation, as $Q = AK^{\alpha}L^{\beta}$. He defines Q as widget output per elapsed time period, K as machine hours used per elapsed time period, and L as labor hours used per elapsed time period. Exponents α and β are pure number (point) elasticities, the percentage changes in output per percentage change in either machine hours or labor hours, respectively. Symbol A "may be either a constant or a variable." Solving for A gives $A = Q/K^{\alpha}L^{\beta}$, which therefore has the dimension units

$$\text{(1a)} \quad \frac{\text{widgets / elapsed-time}}{[\text{machine-hours / elapsed-time}]^{\alpha} \cdot [\text{labor-hours / elapsed-time}]^{\beta}}$$

or

$$\text{(1b)} \quad \frac{\text{widgets} \cdot (\text{elapsed-time})^{(\alpha+\beta-1)}}{[\text{machine-hours}]^{\alpha} \cdot [\text{labor-hours}]^{\beta}}$$

So far, so good—except for a minor quibble: we would describe A, α, and β all as "parameters." A quick-and-dirty definition of "parameter" might be "variable (changeable) constant"—a constant whose value (magnitude) can change, but only exogenously. Therefore, parameters are not *coordinate* variables.[4]

Professor Barnett then argues that: "If $\alpha = \beta = 1$, then the dimensions of K^{α}, L^{β}, and Q . . . are meaningful." But for the dimension of A

> to be meaningful, requires, at a minimum, that the product of machine-hours and man-hours is meaningful, a dubious proposition indeed. [Why?] However, even if the dimensions are meaningful in this case, they are eco-

[3]We do not address the publication issues that Professor Barnett raises in his Addendum and Appendix (pp. 99–104).

[4]A parameter is "an arbitrary constant or a [exogenous] variable in a mathematical expression, which distinguishes various specific cases. . . . Also, the term is used in speaking of any letter, variable, or constant, *other than* the coordinate variables." (James 1968, p. 263; emphasis added). A coordinate is "one of a set of numbers which locate a point in space" (p. 80). Hence a function's coordinate variables are its dependent and independent variables, because the function—*using* its parameters' values and dimensions—*maps* the function's independent variables' domain into the function's dependent variable's range. Only variables get mapped; parameters themselves do not get mapped, because they (together with the function's specific functional form of linear, polynomial, multiplicative, exponential, etc.) help do the mapping.

> nomically unreasonable. For, if $\alpha = \beta = 1$, the marginal products of both K and L are positive constants (the Law of [Eventually] Diminishing Returns is violated) and there are unreasonably large economies of scale—a doubling of both inputs, *ceteris paribus*, would quadruple output. (p. 96)

But if his argument here shows anything, it shows not that the *dimensions* of machine-hours and man-hours are unreasonable, but that his assumed *values* of α and β are unreasonable. If increasing (or constant) returns to either a single input or to scale are unreasonable (especially since the Cobb-Douglas function does not let initial increasing returns switch later to decreasing returns), the solution is to require $\alpha < 1$ and $\beta < 1$. But Professor Barnett does not accept that solution.

His argument continues: "If it is not true that $\alpha = \beta = 1$, then either α or β, or both, have noninteger values or integer values of two or greater. Noninteger values of α or β, or both, result in" roots,

> for example, $(\text{man-hours/year})^{0.5}$ or $(\text{man-hours/year})^{1.5}$ for L^{β}, and similarly for K^{α}. But the square roots of man-hours and of years are meaningless concepts, as are the square roots of the cube of man-hours and the cube of years. Also, integer values of two or greater for α or β, or both, result in such units as . . . $(\text{man-hours/year})^2$ or $(\text{man-hours/year})^3$. . . . [which] are meaningless concepts, . . . and similarly for machine-hours. (The units of A are even more meaningless, if that is possible.) (p. 96)

Professor Barnett surely is comfortable with noninteger *fractional* values of α and β when thinking of them as elasticities (percentage changes). So when the same α and β are roots or powers of man-hours or capital-hours or years, why he sees the results as meaningless is not at all clear. His assertions are not explanations. In the Cobb-Douglas function, $\alpha < 1$ and $\beta < 1$ *roots* are fractional elasticities that are neither unrealistic nor meaningless. Cobb-Douglas $\alpha > 1$ and $\beta > 1$ *powers* do generate invariably increasing returns that are unrealistic (for large input quantities), but not meaningless: we do understand their implications.

Professor Barnett apparently has overlooked that the purpose of *any* function's parameters—be it a function in economics or physics or engineering or pure mathematics or any other discipline—is to help describe the relationship between the function's dependent variable and its independent variables, including their dimension units. Therefore, a function's parameter dimensions (and values) simply *are whatever they need to be* (including roots and powers) *to describe the relationship fully and accurately*—including that the left and right side dimensions must match.

Variables must have (understandable) dimensions. Parameters may or may not have dimensions. If they do not, they are pure numbers (*not* necessarily invariant constants). If they do, their dimensions need not be understandable (although it is nice if they are). Instead, parameter dimensions' *only* requirement is that they *describe the relationship* between the function's dependent and independent variables' dimensions.

In the Cobb-Douglas production function, parameter *A* has dimensions that force the function's left and right side dimensions to match. More generally, parameter *A* has dimensions that the function needs to describe its assumed relationship between output and input variables—for example, to allow the function's α and β parameters to be the percentage change in output that results from a percentage change in real capital or labor input. So unless one rejects the idea that output quantity depends on input quantities, how can one reject the ideas that the percentage change in output quantity could depend on the percentage change of input quantity, and that the percentage relationship could be less than one? Of course, one could reject—quite reasonably—the Cobb-Douglas assumption that output-input elasticities are constant regardless of input quantities, but that is not what Professor Barnett is doing. Instead, he claims that any fractional values for the Cobb-Douglas function's α and β parameters are "meaningless" or "economically unreasonable" simply because they are roots of input quantities. To us, that claim makes no sense, given the role of parameters in defining a production function's relationship between output and input.

2. Cobb-Douglas Parameter Dimensions that are Inconstant

Professor Barnett begins his argument that the Cobb-Douglas function's parameter (*A*, α, and β) *dimensions* are not constant and therefore "nonsensical" (p. 97), by stating that

> this [inconstant dimensions] problem consists in the same constant or variable having different dimensions, as if velocity were sometimes measured in meters per second and other times measured in meters only or in meters squared per second. (p. 97)

He then notes that in Newtonian physics, "a force (*F*) exerted on a body may be measured as the product of its mass (*m*) times its acceleration (*a*); i.e. $F = m \cdot a$, . . . [e.g.,] the units of *F* are kilograms · meters/(second^2)." Then by "Newton's law of universal gravitation," the force of gravity between two objects that are *r* distance apart and of mass *m* and *m′* respectively, can be written $F = G \cdot (mm'/r^2)$, where *G* is the gravitational constant. Solving for *G* gives $G = F/(mm'/r^2)$, so given the units of *F*, *G* has the units

$$\frac{\text{kilograms} \cdot (\text{meters / second}^2)}{\text{kilograms}^2 / \text{meters}^2} = \frac{\text{meters}^3}{\text{kilograms} \cdot \text{second}^2}$$

And: "This result has been invariant for countless measurements of *G* over the past three centuries: regardless of the magnitude [of *G*], the dimensions have always been distance^3/mass · (elapsed time)^2."

We have no problem with any of that. But note that in the function for *G*, the right hand side contains three coordinate variables—the Force attracting the two objects, the Mass of each of the two objects, and the Distance between the two objects—and *no variable parameters*. (The exponents on Mass, on Time, and on Distance are not variable parameters because they cannot vary: they are constant numbers given by the definitions of Force, Mass, and Acceleration. We might denote the exponent on distance as, say, $\alpha = 2$, but there is no point in doing so because the logic of the model does not allow it to vary: either $\alpha = 2$, or the model totally fails.)

Having introduced this gravity model, Professor Barnett compares its

> constancy of the dimensions—with the results of measurements of a 2-input, CD production function. . . . Invariably, alternative estimates of α, β, and A differ. This is not surprising, but . . . because A has both magnitude and dimensions, different values of α and β imply different dimensions for A, such that, even though the dimensions in which Q, K, and L are measured and are constant, the dimensions of A are inconstant. . . . If . . . α and β are measured as 0.5 and 0.5, respectively, then the units of A are wid/(manhr$^{0.5}$ · caphr$^{0.5}$). However, if . . . α and β are measured as 0.75 and 0.75, respectively, then the units of A are wid · yr$^{0.5}$/(manhr$^{0.75}$ · caphr$^{0.75}$). (p. 97)

Actually, however, for all values of α and β, the *dimensions* of A are invariant, and remain defined by (1a) or (1b) above.[5] The exponent on "yr" (elapsed time) *always* is $\alpha + \beta - 1$, the exponent on "caphours" (machine-hours) *always* is α, and the exponent on "manhr" (labor-hours) *always* is β. Different values of α and β change *only the magnitude* of A.

In the gravity model and in the Cobb-Douglas production function model, the estimations or measurements are very different conceptually. In the gravity model, regardless whether its left-side variable is F or G, *all* symbols other than G are known values; the *only* unknown is G. In the Cobb-Douglas model, the magnitudes of A, α , and β *all* are unknown, and are estimated as a "best fit" to known data for output quantity q and for input quantities K and L. And while the gravity model applies to the theoretically identical quantitative behavior of objects in one universe, the Cobb-Douglas model—despite some severe nondimensional theoretical limitations (which we mention later)—has been used (either heroically or recklessly) to describe the behavior of widely disparate enterprises: single firms in entirely different industries (peanut farming, . . . internet services, . . . pharmaceuticals invention and production; nonprofit private organizations; local, state, and national governments—and price-weighted aggregate data for multi-output enterprises, entire industries, and even entire economies.[6]

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[5]De Jong writes (1967, p. 19, n. 1): "The dimension of a certain variable tells us how the numerical value of that variable changes when the units of measurement are subjected to changes." The same statement would apply to parameters.

[6]Of course, as has long been understood (Baumol 1977, pp. 350–53; see also chaps. 11, 24), price-weighted aggregate quantity data raises serious practical and ultimately insoluble theoretical and logical difficulties, as Professor Barnett points out (p. 96, n. 7).

Moreover, even in what may appear to be the "same" situation, say when a particular firm's output of a particular product changes over time, it is *normal* for "same situation" production function parameter estimates to change over time. Production processes are the result of human knowledge and decisions. Humans are not Pavlov's dogs, but acting beings, and they develop new knowledge and forget old knowledge even when it is useful.

Economic relations change as individual understanding of those relations changes. In contrast, the world of Newtonian physics does not change (at least not rapidly enough to notice without incredibly precise instrumentation), and it does not change *because* people come to understand it better. Gravity now works as gravity did before Newton was hit by the apple, and as it did millions of years ago.

If the elasticity of output with respect to capital or labor input were even approximately constant for all the products that have been studied using the Cobb-Douglas (or *any* specific) production function, or constant over substantial periods of time for the same product, *that* would be amazing—and rather than a cause for rejoicing, it would be cause for suspecting that someone was cooking either the estimating algorithm or the data or both.

When Professor Barnett compares the absurdity of measuring velocity "sometimes in meters per second and other times in meters only or in meters squared per second" to estimating different values for A, α, and β, he forgets that we *define* velocity as meters per second and acceleration as meters per second squared because any other dimensions would be logically wrong and would make no sense. In acceleration, the "2" exponent on "seconds" is an invariant constant that comes from the meaning of acceleration: change in the rate of change; change in meters per second, per second. (A less obvious version of that statement is that the "2" results from a mathematical operation: taking the derivative of velocity's definition, with respect to time.) But in specific economic functions, including production functions, parameters such as A, α, and β are *not* defined invariant constants: we *measure* them as *variable* parameters that *of course* change to fit different situations and different time periods, as they are *expected* and *supposed* to do.

Specific production functions are not part of the realm of pure economic theory, but are tools of historical analysis. To demand constancy for a production function's parameter values simply makes no economic sense.

Professor Barnett's analysis has not shown us any dimensional errors in the Cobb-Douglas production function.[7] However, the Cobb-Douglas function *does* have severe *non*dimensional limitations.[8]

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[7]For a much more formal and "philosophical" analysis of Cobb-Douglas (and constant elasticity of substitution) type functions, see De Jong (1967, pp. 34–50; for noninteger exponents, pp. 46–50). Our discussion of the Cobb-Douglas function has treated it as a "*fundamental equation*," analyzed using the "*traditional method*" (pp. 34–37).

[8]We discuss these nondimensional limitations in Appendix A.

3. Macroeconomic Example

After his Cobb-Douglas discussion, Professor Barnett turns to a macroeconomic model (pp. 97–98), cited only as an unspecified paper from "a recent issue of a leading English-language economics journal." He does not tell us much of what the model is about.

His quotations from that paper do tell us that the model has a representative household whose complete present utility function *includes* the sum of an infinitely long stream of discounted future per-period work-utility functions, $H(N_t,U_t)$, each dependent on that period's hours worked N_t and also on work effort U_t.

The model has also "a continuum of firms distributed equally on the [closed] unit interval, . . . indexed by $i \in [0,1]$" (p. 97). Since any continuum between any two points on a line contains an infinite number of points, this model has an infinite number of firms (p. 98).

Also, each firm "produces a differentiated good with a technology [production function] $Y_{it} = Z_t L_{it}^{\alpha}$. L_i may be interpreted as the quantity of effective labor input used by the firm, which is a function of hours and effort: $L_{it} = N_{it}^{\theta} U_{it}^{1-\theta}$ where $\theta \in [0,1]$."[9] And "Z is an aggregate technology index [apparently common to all firms], whose [random] growth rate is assumed to follow an independently and identically distributed (i.i.d.) process." There is a bit more detail about the Z_t technology index,[10] but we have enough for Barnett's criticisms of this model.

Professor Barnett's last quotation describing the model is that

> "in a symmetric equilibrium all firms will set the same price P_t and choose identical output, hours, and effort levels Y_t, N_t, U_t. Goods market clearing requires . . . [Barnett's ellipses] $Y_{it} = Y_t$, for all $i \in [0,1]$, and all t." Furthermore, the model yields "the following reduced-form equilibrium relationship between output and employment: $Y_t = AZ_t N_t^{\phi}$." (p. 98)

Barnett's Critique of the Macroeconomic Model

For this model, Professor Barnett has three major criticisms,

> conclusions that can be drawn from this model. . . . (1) [T]he number of firms and the number of households is identical, and is equal to infinity; (2) the quantity of each input used by each firm is identical to the quantity of each input provided by each household; and, (3) there are an infinite number of differentiated goods, each of which is identical to every other good.

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[9]Although here θ is defined over a closed interval, $\theta \in [0,1]$, Barnett's footnotes 15 and 16 (p. 98) define θ over an open interval, $\theta \in (0,1)$; we think the notes probably are correct.

[10]The process is "$\{\eta_t\}$, with $\eta_t \sim N(0, s_z^2)$. Formally, $Z_t = Z_{t-1} \cdot exp(\eta_t)$." Barnett, p. 97, and also his footnote 15 (p. 98). That is, although in any period Z_t has a given value (equal for all firms), from period to period its value varies randomly: each Z_t equals the preceding period's Z_{t-1} times an exponential function of η_t, which follows a normal distribution with mean of 0 and variance of s_z^2.

He notes that item (2) does *not* imply that each firm's inputs are supplied by only one household (p. 98).

Given a continuum of firms, the "number" of firms *is* infinite But trouble arises when he assumes that the infinite number of firms is the number n, and then calculates that since (in symmetric equilibrium) each firm uses N_t labor hours and U_t labor effort, the total labor hours and labor effort used are $\mathrm{n}N_t$ and $\mathrm{n}U_t$. Given that N_t and U_t also represent the amount of labor and effort supplied by a single representative household, he concludes that the infinite number of firms must equal the infinite number of households, or else there would be excess demand or supply for labor hours and effort[11] in this equilibrium.

This conclusion is invalid, because it relies on "infinite" being an integer number. But "infinite" is not a number, and neither is "infinity."[12] Moreover, a continuum includes both integer *and* irrational real numbers, and contains an *uncountable* (nondenumerable) infinity (distinguished from a *countable* [denumerable] infinity) of points.[13] So a "continuum of firms" does not imply a countable number of firms to which the number of households can be compared. For Barnett's first "conclusion" about this model, that the number of firms equals the number of households, his reasoning fails.

For his second "conclusion" about this model, that the inputs (labor-hours and labor-effort) used by each firm equal the inputs supplied by each household, his reasoning fails again, because it depends on the number of firms and households being equal. But his second "conclusion" does tend to be supported by the model's use (in symmetric equilibrium) of the same symbols (N_t

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[11]In Barnett's words (p. 98): "Assume, *arguendo*, that the (infinite) number of firms is given by n. Then . . . the total hours used [by firms] is $\mathrm{n}N_t$ and the total effort level used is $\mathrm{n}U_t$. . . . [U]nless there are exactly n households providing $\mathrm{n}N_t$ total hours and $\mathrm{n}U_t$ total level of effort, either the firms are using more hours than the households are actually working, or they are using less. The same can be said for the level of effort."

[12]From Courant and Robbins (1969; all emphasis in the original):

> It is, however, sometimes useful to denote such expressions [created by (taking the limit of) something divided by zero] by the symbol ∞ (read, "infinity") *provided that one does not attempt to operate with the symbol ∞ as though it were subject to the ordinary rules of calculation with numbers*. (p. 56) . . . The sequence of all positive integers . . . is the first and most important example of an infinite set. . . . But in the passage from the *adjective* "infinite," meaning simply "without end," to the *noun* "infinity," we must not make the assumption that "infinity" . . . can be considered as though it were an ordinary *number*. We cannot include the symbol ∞ in the real number system and at the same time preserve the fundamental rules of arithmetic. (p. 77)

[13]From Courant and Robbins (1969, pp. 79-83, see also pp. 77-78; all emphasis in the original), "The Denumerability of the Rational Numbers and the Non-Denumerability of the Continuum": "*The set of all real numbers*, rational and irrational, *is not denumerable*. In other words, the totality of real numbers presents a radically different and, so to speak, higher type of infinity than that of the integers or of the rational numbers alone" (p. 81).

and U_t) for each firm's input use and for the representative household's input supply. That notation puzzles us a bit.

His third "conclusion" about this model has two components: first, that the number of goods is infinite; second, that the goods are differentiated yet identical. The first component does follow directly from the model's assumptions of a "continuum of firms" each producing only one good, but the "infinite" number of firms and goods is uncountable. The second component's claim that the goods are differentiated yet identical is difficult to understand, particularly without reading the original paper. One reconciliation would be for the goods to be differentiated without using different production processes—for example, different color but otherwise identical bicycles. Our preferred hypothesis—which is consistent with where the words "differentiated" and "identical" appear in Barnett's quotations from the original paper—is that in disequilibrium, each firm produces a differentiated product, and then symmetric equilibrium forces all firms to produce identical outputs, produced using identical inputs and selling at the same price.

Barnett's own argument for his third "conclusion" is entirely different (p. 98). It results from his dimensional analysis of the model's output and employment: $Y_t = AZ_tN_t^{\theta}$ (discussed in our next section). He argues that

> because . . . A and Z_t are both dimensionless magnitudes, Y_t must have the same dimensions as N_t^{ϕ}. The dimension of N_t is hours; and ϕ is a positive, dimensionless, constant. Therefore, the dimensions of Y_t are hrs$^{\phi}$. . . . [If] $\phi \neq 1$, the dimension of Y_t . . . is meaningless.[14] If $\phi = 1$, . . . the dimension of Y_t is the same as that of N_t, hrs. However, in that case, . . . the output hours are less than, equal to, or greater than the input hours as AZ_t is less than, equal to, or greater than one (1). But if output is measured in hours, then *the output hours cannot be greater than or less than the input hours*; i.e., $AZ_t \equiv 1$ and $Y_t \equiv N_t$. . . . [*T*]*here is no net production*. . . . [E]ach of the n differentiated goods produced by the n firms consists of homogeneous hours. Surely, this model is not defensible. (p. 98; emphasis added)

Thus Professor Barnett's dimensional analysis extends his third conclusion, from firms producing identical outputs to firms producing either meaningless outputs, or else homogeneous outputs all measured in hours, with the devastating consequence of no net production.[15] However, we are not convinced that Z_t and A are dimensionless pure numbers.

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[14]This apparently is a reprise of his argument against the Cobb-Douglas production function (in our main text above, on page 48), that roots and powers of economic variables and parameters are meaningless.

[15]Here is an aside that may, perhaps, misrepresent Professor Barnett's point: Even if both sides of a production function have dimension units of elapsed time, we see no problem regardless whether output hours are less than, equal to, or greater than input hours. The issue should be whether net *value* production occurs, and that depends not only on the quantities but also on the values (prices) of the outputs and inputs. Our guess is that for any airline, output time of passenger flight hours conceivably is less than input hours (for pilots, flight attendants, baggage handlers, reservations and boarding staff, and especially maintenance personnel, and don't forget air traffic control). But in any case, net value production can and does occur.

Conventionally Calculated Dimensions

Before examining Professor Barnett's dimensional analysis of this model, first consider the conventionally calculated dimension units for each firm's technology or production relationships, using the previously stated principle that a function's parameter values and dimensions simply *are whatever they need to be* to describe the relationship.

In the technology function $Y_{it} = Z_t L_{it}{}^{\alpha}$, substituting $L_{it} = N_{it}{}^{\theta} U_{it}{}^{1-\theta}$ gives $Y_{it} = Z_t[N_{it}{}^{\theta} U_{it}{}^{1-\theta}]^{\alpha} = Z_t[N_{it}{}^{\alpha\theta} U_{it}{}^{\alpha(1-\theta)}]$. Let the firm's output Y_{it} be widgets per elapsed time period, N_{it} be labor-hours per elapsed time period, and U_{it} be labor-effort per elapsed time period. We agree with Barnett (footnote 16, p. 98) that exponents α and θ (and also ϕ, used in our next paragraph) are "positive, dimensionless, constants" (pure numbers). But we think that to match the dimensions of both sides of the firm's technology or production relationship, Z_t must have dimension units

$$\text{(2a)} \qquad \frac{\text{widgets / elapsed-time}}{[\text{labor-hours / elapsed-time}]^{\alpha\theta} \cdot [\text{labor-effort / elapsed-time}]^{\alpha(1-\theta)}}$$

or

$$\text{(2b)} \qquad \frac{\text{widgets} \cdot (\text{elapsed-time})^{(\alpha-1)}}{[\text{labor-hours}]^{\alpha\theta} \cdot [\text{labor-effort}]^{\alpha(1-\theta)}}$$

In the reduced-form solution for symmetric equilibrium, all firms produce the same output quantity (which allows dropping the subscript identifying the *i*th firm): $Y_t = AZ_t N_t{}^{\phi}$. Before worrying about what A and ϕ are or where they come from, consider the conventionally calculated dimension units for this equilibrium relationship. Y_t again is widgets per elapsed time period, and N_t again is labor-hours per elapsed time period. And ϕ is a pure number (as in Barnett; see our preceding paragraph). Then to match the dimensions of both sides of this relationship, A must have dimension units

$$\text{(3a)} \qquad \frac{\text{widgets / elapsed-time}}{\dfrac{[\text{widgets} \times (\text{elapsed-time})^{(\alpha-1)}] \cdot [\text{labor-hours / elapsed-time}]^{\phi}}{[\text{labor-hours}]^{\alpha\theta} \cdot [\text{labor-effort}]^{\alpha(1-\theta)}}}$$

or

$$\text{(3b)} \qquad [\text{labor-hours / elapsed-time}]^{(\alpha\theta-\phi)} \cdot [\text{labor-effort / elapsed-time}]^{\alpha(1-\theta)}$$

or

$$\text{(3c)} \qquad [\text{labor-hours}]^{(\alpha\theta-\phi)} \cdot [\text{labor-effort}]^{\alpha(1-\theta)} / (\text{elapsed-time})^{(\alpha-\phi)}$$

But as noted above, Professor Barnett would not accept these dimension calculations for either Z_t or A. For him, "A and Z_t are both dimensionless magnitudes" (p. 98, n. 15).

He states that "Z_t must be a positive, dimensionless, variable because 'it is an aggregate technology index'" (given its description copied and referenced in our footnote 10). But that description simply defines the probabilistic change of Z_t from Z_{t-1}. Probabilistic change does not mean that the variable that is changing must be dimensionless.

So we reject Professor Barnett's claim that technology index Z_t is dimensionless.

In general, *an index need not be dimensionless.*[16] An example of a dimensioned index is in Irving Fisher's equation of exchange ($MV = PT$). Each side's dimension must be the flow of dollars per time period (as Professor Barnett correctly implies, p. 96, n. 8). But to get that to happen, price index P and transactions quantity index *T cannot both be pure dimensionless numbers*. Either one *or* the other must be dimensioned, as we explain in Appendix B.

As for parameter A, Professor Barnett writes

> we are given that: $A \equiv [\lambda_n(1\text{-}\theta)/\lambda_u\theta]^{\alpha(1\text{-}\theta)/(1+\sigma u)}$; . . . and, λ_n, λ_u, σ_n, σ_u are positive constants. . . . [They] are dimensionless from the context in which they first appear: $H(N_t, U_t) = (\lambda_n N_t^{1\text{-}\sigma n} / (1+\sigma_n)) + (\lambda_u U_t^{1\text{-}\sigma u} / (1+\sigma_u))$. . . . [And α and θ are pure, dimensionless, constants, as we agree.] Therefore, A must be a positive dimensionless constant. (p. 98, n. 15)

Our analysis here is handicapped by not knowing why $H(N_t, U_t)$ has this specific functional form, what λ_n, λ_u, σ_n, and σ_u really are, and why Professor Barnett thinks that *all* these positive constants need be pure dimensionless numbers. Nevertheless, Professor Barnett's argument does not persuade us. Consider the preceding paragraph's last equation. On the left side, $H(N_t, U_t)$ "measures the disutility from work" (recall Barnett's first quotation from his source), which suggests that the left side's dimension is some measure of disutils, or else a pure number that ranks less preferred combinations of labor

[16]De Jong (1967, pp. 23–24) writes:

> Are not index numbers dimensionless products? The answer is: this may well be so, but not necessarily. *The answer "yes" or "no" depends upon what is best adapted to the problem in hand*. It is essential to realize clearly that no simple "cookery book recipes" exist for this; the only good guidance is a consideration of the setting of the *economic* problem one wishes to analyze. For instance, if the economic problem is such that we are just interested in the *ratio* between absolute prices at two points of time, t' and t_o', nothing prevents us from considering this ratio as a dimensionless entity. On the other hand, we may use dimensional analysis as a device for checking an economic equation . . . , [which] invites us to assign a dimension to every variable capable of such an assignment, even if it happens to be a price index number; otherwise, no dimensional check would be possible. (Emphasis in the original)

In this quotation, De Jong is discussing the Equation of Exchange. See also De Jong's discussion (pp. 33–34) of a Gardner Ackley wage index problem, his discussion (pp. 6–23) of basic primary and secondary dimension concepts, and his reference to Bridgman (p. 24, n. 1).

hours and effort—which Barnett (2003, pp. 42, 46, 48–55) might prefer to "disutils." On the right side, the only symbols that Barnett does not claim to be dimensionless are N_t and U_t, which respectively have dimension units of labor-hours and labor-effort per elapsed time period. So *something* else on the right side—perhaps λ_n and λ_u—must have dimensions that "convert" labor-hours and labor-effort per elapsed time period into either disutils or a pure number. If so, A cannot be dimensionless.

Given the available information, we see no reason to abandon our dimensions for Z_t given in our (2a) and (2b), or our dimensions for A given in our (3a) and (3b). We see no dimensional problems in the macroeconomic model discussed by Professor Barnett.

Final Thoughts

Economists may not pay much attention to dimensional analysis, but that does not mean that unstated but implied dimensions and units are wrong or invalid. To us, Professor Barnett has not demonstrated his specific allegations of dimensional errors. And he has not persuasively shown any serious problem—much less a fatal flaw—in contemporary mathematical economics, *caused by dimensional errors*.

But our critique of Barnett's diagnosis does not imply that we judge the mathematical economics patient to be healthy. In contemporary mainstream economics, there is plenty of misuse and abuse of mathematics that has nothing to do with dimensional errors: a propensity to disregard fundamental elements of economic reality simply because they cannot be encapsulated in mathematical models; the misuse of mathematics to foster the illusion that economists can provide decision makers with information that no individual mind can possess; and disguising normative judgments as being positive conclusions of so-called welfare economic analysis (Boettke 1997 and Rothbard 1956).

We do not share, however, the implacable hostility of some Misesians to the use of mathematics in economics. Some economists find that mathematics gets in the way of their thinking, in which case they shouldn't use mathematics in their thinking process. Other economists find that mathematics helps them to see variables and effects that they may otherwise overlook; to better distinguish endogenous from exogenous variables;[17] and to avoid deductive errors.[18] After clarifying their thoughts, sometimes they can follow

[17]It is easy to go astray in verbal-logical analysis by improperly treating as exogenous, a variable that is endogenous: for example, implicitly treating (real) income as an exogenous variable while deriving the labor supply response to a wage change. See Gonzalez (2000).

[18]Those same economists can, of course, find mathematics not useful for thinking about some issues, despite finding it useful for other issues.

Alfred Marshall's famous advice to "Burn the mathematics."[19] But on other occasions they should publish rather than burn the mathematics: a lot of obscurity, misinterpretation, and semantic debates can be avoided by communicating some ideas with the assistance of some mathematics.

Moreover, economics is more than economic theory. Most economists want to address issues that require going beyond economic theory. Since economic theory's laws always are qualitative, never quantitative, *no* reasonable estimate or even educated guess can be made about the *magnitude* of *any* economic effect, *unless* one is willing to go beyond the realm of pure economic theory. With theory alone, for example, one cannot assess the employment effect of a twenty percent hike in the Federal minimum wage. In fact, and contrary to what some economists incorrectly believe, one cannot derive even the *sign* of the employment effect from only the pure Misesian logic of choice.[20]

Specific economic functions simply are tools of historical analysis. Their usefulness in applied economics cannot be decided *a priori*; it must be judged by how well they perform their intended use.

The extent to which mathematics can assist the Austrian research program is an empirical issue. But there is no reason for Austrians to fear mathematics properly used. The *rejection* of mathematics is neither necessary nor sufficient for doing *good* economics. And the *use* of mathematics is neither sufficient nor necessary for doing *bad* economics. What distinguishes Austrian economics from bad economics is the Austrian theoretical hardcore.

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[19]In 1906, Alfred Marshall wrote:

> I had a growing feeling in the later years of my work at the subject that a good mathematical theorem dealing with economic hypotheses was very unlikely to be good economics: and I went more and more on the rules—(1) Use mathematics as a shorthand language, rather than as an engine of inquiry. (2) Keep to them until you have done. (3) Translate into English. (4) Then illustrate by examples that are important in real life. (5) Burn the mathematics. (6) If you can't succeed in (4), burn (3). This last I did often.

Favorably quoted by many economists, for example by Colander (2001, p. 131), citing the following source: "From a letter from Marshall to A.L. Bowley, reprinted in A.C. Pigou, Memorials of Alfred Marshall, p. 427."

[20]The Misesian logic of choice tells us that employers will not willingly hire labor whose marginal hiring cost exceeds its marginal benefit. It does not tell us, however, what counts as a hiring cost and what counts as a hiring benefit in employers' minds, or whether the affected labor markets are competitive or monopsonistic.

APPENDIX A
THE COBB-DOUGLAS PRODUCTION FUNCTION'S LIMITATIONS

In its traditional form, the Cobb-Douglas production function is written $Q = AK^{\alpha}L^{\beta}$, where Q is output, and K and L are real capital and labor inputs, respectively. It can also be written more generally, for one output q produced by I inputs x_i, as:[21]

$$q = f(x_1, x_2, x_3, \ldots x_i, \ldots x_I) = ax_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3} \ldots x_i^{\alpha_i} \ldots x_I^{\alpha_I}$$

But in either its traditional or more general form, this specific multiplicative production function has several seriously unrealistic characteristics.

(a) Because it is multiplicative, if any input quantity is zero, output quantity is zero, no matter how many other nonzero input variables there are, or how large their quantities are. Every input is essential.

(b) Because it has only one term, it cannot exhibit positive followed by negative returns—if using more of a single input initially increases output quantity, using much more of that input cannot decrease output quantity: in a Cobb-Douglas world, excess fertilizer never could burn plants enough to decrease corn output.[22]

(c) Because it is multiplicative with only one term, returns to a single variable input (holding all other inputs constant) *cannot* change; they always will be qualitatively the same. In a Cobb-Douglas production function,

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[21]In a Cobb-Douglas type (single-term multiplicative) production function, the initial coefficient (A or a_0) and the input exponents (α and β or α_i) usually are positive, so that output increases as the ith input increases. However, if the production process must cope with production impediments (cotton production in a world containing boll weevils comes to mind), the impeding variable's exponent would be negative.

For *any* single-output multiple-input production function, output quantity q and usually all input quantities x_i are defined as flows (for example, flows of raw materials and of labor and real capital services) per time period rather than as stocks at a point in time. But for inputs such as dirt in a production function for housing services or perhaps for agriculture, or catalyst inputs in an oil refinery, the appropriate input dimension could be an unchanging stock measured at a point in time.

[22]However, a Cobb-Douglas type production function can be generalized by adding a second term:

$$q = f(x_1, x_2, x_3, \ldots x_i, \ldots x_I) = a_0x_1^{\alpha_{01}}x_2^{\alpha_{02}}x_3^{\alpha_{03}} \ldots x_i^{\alpha_{0i}} \ldots x_I^{\alpha_{0I}} + a_1x_1^{\alpha_{11}}x_2^{\alpha_{12}}x_3^{\alpha_{13}} \ldots x_i^{\alpha_{1i}} \ldots x_I^{\alpha_{1I}}$$

If $a_1 < 0$, negative returns to single inputs become possible. This two-term multiplicative function remains homogeneous of degree h (see our note 25), if $h = \Sigma\alpha_{0i} = \Sigma\alpha_{1i}$, where the summations are $i = 1, 2, 3, \ldots I$.

> increasing the variable input quantity *always* will increase output quantity at *either* an increasing, a constant, *or* a decreasing rate. As the one variable input increases, output quantity *never* will switch from, say, increasing at an increasing rate to increasing at a constant or decreasing rate, no matter how small or large the single variable input quantity is.[23] If diminishing returns occur, they must occur not only eventually but also initially, as soon as the variable input quantity begins to increase from zero. (Different inputs can, however, have qualitatively different returns: for example, if output increases at an increasing rate as input *i* increases, output may increase at a decreasing rate as a different input *j* increases—in each case changing only one input quantity and holding all other inputs constant.) In more general production functions,[24] "returns" to a single variable input *can* change: as a single variable input quantity increases, the rate at which output increases can switch among increasing, constant, or decreasing.

And because the Cobb-Douglas function is homogeneous,[25] it cannot have inflection points at which the relationship between output quantity and two or more input quantities can switch among convex, linear, or concave, and it can have no freestanding additive constant. (The "homogeneity" name comes

[23]For *any* production function, whether a single variable input increases output at an increasing, constant, or decreasing rate is determined by the sign (positive, zero, or negative) of the function's second partial derivative with respect to that input. But for a multiplicative function, returns to a single input cannot vary qualitatively, because the sign of the second partial derivative with respect to an input will always be either positive, zero, *or* negative (depending on whether the input's exponent parameter's sign is greater than, equal to, or less than 1), regardless of the variable input's quantity. [The statement in parentheses assumes $f(\cdot) > 0$.]

[24]Including some homogeneous production functions. (For homogeneity, see our next note). For specific examples, see the string of thirteen *American Economic Review* Communications on "Diminishing Returns and Linear Homogeneity," from Nutter (1963) to Piron (1966) and Eichhorn (1968). For the complete list, see their references. Also relevant is Levenson and Solon (1966).

[25]To define homogeneity, first consider a general (not necessarily homogeneous) function $q = f(x_1, x_2, x_3, \ldots x_i, \ldots x_I)$, with $q_o = f(x_{1o}, x_{2o}, x_{3o}, \ldots x_{io}, \ldots x_{Io})$ for any set of specific $x_i = x_{io}$ values. This function is homogeneous of degree h if and only if multiplying all x_{io} values by η gives $q = \eta^h q_o = \eta^h f(x_{1o}, x_{2o}, x_{3o}, \ldots x_{io}, \ldots x_{Io}) = f(\eta x_{1o}, \eta x_{2o}, \eta x_{3o}, \ldots \eta x_{io}, \ldots \eta x_{Io})$. That is, a function is homogeneous of degree h if and only if η^h can be "factored completely out" of the entire function, when within the function each independent variable is multiplied by η. Any Cobb-Douglas type (single-term multiplicative) function always satisfies this condition, with $h = \alpha + \beta$ or $h = \Sigma\alpha_i$.

In contrast, a polynomial function is homogeneous of degree h only in the special case that in every term, its variables' exponents sum to h. To convert an ordinary polynomial to homogeneity of degree h, the coefficients of all terms whose variables' exponents do not sum to h must be set to zero (which, assuming $h > 0$, includes setting the additive constant term to zero). Homogeneity is very restrictive. An ordinary polynomial, no terms forced to zero, third degree (cubic) or higher to allow inflections between convex and concave, usually is a much more realistic theoretical representation of almost any production relationship.

from these "same shape" characteristics.) All homogeneous functions have the following "returns to scale" properties:

(d) Holding only some (or no) input quantities constant, with an initial mix of at least two variable inputs, returns to scale for that mix of variable inputs cannot change: increasing all variable input quantities together (in fixed proportions) *always* will increase output quantity at *either* an increasing, a constant, *or* a decreasing rate. As the variable inputs increase ("scale up") together, output quantity *never* will switch from, say, increasing at an increasing rate to increasing at a constant or decreasing rate. For a given set of constant inputs, returns to scale for the variable inputs always will be qualitatively the same, no matter how small or large the fixed-proportion package of variable inputs is.[26]

(e) Again holding only some (or no) input quantities constant, changing the initial mix of the variable inputs does *not* change whether scaling up all of the variable input quantities increases output at an increasing, constant, or decreasing rate. That is, for a given set of constant inputs, returns to scale always will be qualitatively the same—increasing, constant, *or* decreasing—no matter what mix of variable inputs is "scaled up" in fixed proportions.[27]

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[26]For *any* production function, to determine its returns to scale, first replace each of its fixed inputs x_i by $x_i^{\#}$ (the superscript $^{\#}$ denotes a constant value variable), and replace each of its variable inputs x_j that will change only in fixed proportion by ηx_{jo}, where η is the "scale factor" that will determine how much of the initial mix of variable inputs will be used. Then the sign of function's second partial derivative with respect to η (holding the $x_i^{\#}$ and x_{jo} constant) will give the function's returns to scale, which in general may vary among increasing, constant, or decreasing, as η increases. But in any homogeneous function, including the Cobb-Douglas function, this second derivative's sign is constant (either positive, zero, or negative), not affected by the magnitude of η.

Geometrically, consider a three-dimensional two-variable-input production function in the corner of a room. Input space is on the floor. The horizontal x_1 and x_2 input axes each are where the floor meets a wall. The vertical q output quantity axis is where the two walls intersect. In input-space on the floor, the origin at the corner of the room, together with the point defined by x_{1o} and x_{2o}, define a ray. Changes in the value of scale factor η move along that ray. Above that ray, in output space, is the output quantity q generated by different values of η. As η increases, output increases at either an increasing, constant, or decreasing rate of return to scale. But if the function is homogeneous, the rate of return to scale never switches from one curvature to another.

[27]To demonstrate this property for any homogeneous function, including the Cobb-Douglas function, note from the preceding note's calculations that the sign of the second partial derivative of the function with respect to η is not affected by changes in any of the variable input quantities x_{jo} [assuming $f(\cdot)$ remains $f(\cdot) > 0$]. Geometrically, in the preceding note's three-dimensional room-corner diagram (of a homogeneous production function), above *any* ray on the floor, as η increases, output always increases at either an increasing, constant, *or* decreasing rate of return to scale, regardless of the ray considered.

(f) And a homogeneous function (of degree $h > 0$) can have no additive constant term, although such a term (if positive) allows for some minimum output quantity to be provided by nature, or (if negative, together with a nonnegativity condition for the function's output quantity) allows input quantities to reach some minimum before any output is produced.

Professor Barnett's "Dimensions and Economics" paper (2004) does not deal with any of these limitations.[28] He focuses his attention on variable and parameter dimensions (pp. 95, 96–98).[29]

Appendix B
Fisher's Equation of Exchange, A Dimension For Either P or T

In Irving Fisher's equation of exchange ($MV = PT$), the left side is the dollar value of the flow of spending per time period; the right side is the dollar value of the flow of transaction receipts per time period [what might be called nominal "Gross Gross Domestic (or National) Product," since Fisher's equation includes intermediate transactions among businesses (and, by extension, among businesses and governments and between both)]. The economy's gross spending equals its gross receipts, in dollars (or any other monetary unit). The dimension units on both sides—dollars/time-period—must match.

[28]In an earlier paper, Professor Barnett (2003) does point out two of the above Cobb-Douglas function characteristics: (a) for production functions and (d) for utility functions (p. 47, notes 12 and 13 respectively).But we disagree with his claim (note 13) that for a utility function, "a scale increase of the arguments [that] gives rise to a greater scale increase in utility . . . [does not accord] with human action." That statement refers to cardinal (*vice* ordinal) values. It *is* true (as Barnett says) that any particular utility function's dependent variable values, independent variable values, and the *results* that describe consumer choice actions, all are *cardinal*. But those cardinal results (behavior and demand function descriptions) depend *only* on the utility function values' *ordinal* property (larger values represent preferred bundles). For successively preferred bundles, successive increments of larger utility values are arbitrary—they can be of any positive size. Thus *any* increasing monotonic transformation of a utility function has *no* effect on the function's *results* that describe consumer choices: its first finite difference quotient (in the limit, derivative) with respect to scale must be positive, but its second and higher finite difference quotients (or derivatives) can be of any sign. (Derivatives, of course, require continuity, a simplifying but often suspect assumption, as Barnett argues on pages 57–59.)

[29]Professor Barnett does say (2004, p. 95, n. 5), that because a *function's* dependent variable must be unique for any set of independent variable values, "it is incorrect to express any production relationships [as functions?] in any case in which Leibensteinian style X-inefficiency can exist." (See also the macroeconomic model on his page 97, including note 12.) We would rather say that the traditional production function describes the *maximum* output quantity—not the feasible set of output quantities—that the firm can obtain from a given set of input quantities. Given such a production function, to describe not the maximum output quantity but instead the feasible set of output quantities, convert the production function *equation* into a *relation*, by replacing the = sign by a ≤ sign.

On the left side, M is the money stock at a point in time; V is the turnover of that money stock per time interval (e.g. 30 per year); MV is the stock of dollars times 30/year, which is the flow of dollars spent per year. (After stripping out the magnitudes of these variables, M's dimension unit is simply dollars; V's dimension unit is 1/year.) So far, no problem.

On the right side, things are not quite that simple. It is tempting to define price index P as a dimensionless (pure) number ratio of the sums of (quantity weighted) prices for two time periods, and transactions quantity index T as a dimensionless (pure) number ratio of the sums of (price weighted) transaction quantities for two time periods. But if we do that, then multiplying the two dimensionless numbers P and T gives another dimensionless number, rather than a flow of gross receipts of dollars per year.

Therefore, in the equation of exchange, price index P as a dimensionless pure number ratio and transactions quantity index T as a dimensionless pure number ratio, *cannot hold simultaneously*. The solution is that either one or the other ratio must be multiplied by the flow of nominal transaction quantities during the base time period, thus making either P *or* T a dimensioned index.[30]

That conclusion applies not only to Fisher's equation, but also to a "nominal income" equation of exchange ($MV = PQ$), in which the right side does not include intermediate transactions but instead is nominal Gross Domestic (or National) product [Q or sometimes Y being *real* Gross Domestic (or National) product], and Velocity is a substantially smaller number than in Fisher's equation.

A more detailed and more symbolic version of this argument[31] follows.

Let Δt be the length of a time period, in whatever units time is measured: if time is measured in years with 52 weeks per year, then a thirteen week time

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[30]Barnett (2004, pp. 100, 101) mentions that a referee for "a leading English-language economics journal" wrote Barnett that

> Dimensional analysis can only be applied to *laws*. . . . Fisher's relation of exchange . . . $MV = PT$ . . . is one of the few examples that comes closest to a law. One result of dimensional analysis is that there is something odd with this equation. The left part does contain a time dimension, while the right side doesn't. This is not something new and can be found in any textbook.

We see no reason for dimensional analysis to be limited to "laws," and we wonder how the referee would define them, and we'd like to see a representative "any textbook," but that's not why we have quoted that referee.

Instead, we suggest that not seeing the right side's time dimension may result from assuming that P and T both are dimensionless pure ratios. The right side's time dimension comes from multiplying either the price or transaction ratio by the *flow* of nominal transaction quantities *during* the base period.

[31]Based on Boulding (1966, pp. 27–28); and loosely on De Jong (1967, pp. 23–30).

period is $\Delta t = 0.25$ years. A time interval equals or is some multiple of the time period Δt.

On Fisher's left side, M is the money stock at a point in time (at a time interval's end; a more general formulation would allow the point in time to be anywhere in the interval), with its dimension units simply being dollars. Velocity is the money stock's average turnover per time period (during the interval), so it is a number (say 20 times per year, or 5 times per quarter), with dimension units 1/period. Multiplying M by V gives left side dimension units of dollars/time period.

On Fisher's right side, P is a price index, and T is a transactions quantity index. Let p_j and q_j represent the price and quantity of the jth transaction, and superscripts ^{b} and ^{e} respectively represent the base period and end period within the interval. Assume P is a Laspeyres index (base period quantity weights). Then the end period dollar value of the flow of transaction receipts per time period is: $PT = \Sigma p_j^e q_j^e$, where

$$P = \Sigma p_j^e q_j^b \,/\, \Sigma p_j^b q_j^b, \text{ and}$$
$$T = [\Sigma p_j^e q_j^e \,/\, \Sigma p_j^e q_j^b]\cdot\Sigma p_j^b q_j^b.$$

The transactions index T, rather than being a pure ratio, has been redefined into a dimensioned index.

Alternatively, the Laspeyres price index P could be redefined into a dimensioned index as $P = [\Sigma p_j^e q_j^b \,/\, \Sigma p_j^b q_j^b]\cdot\Sigma p_j^b q_j^b = \Sigma p_j^e q_j^b$, so that T would be a pure ratio.

Analogous results hold if P is a Paasche price index (end period quantity weights), and T is defined accordingly (the final factor of either T or P again is $\Sigma p_j^b q_j^b$).

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