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ARTICLES

A Formal Model in Hayekian Macroeconomics: The Proportional Goods-in-Process Structure of Production

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The core concept of Austrian macroeconomics is the structure of production, which makes it possible to analyze the intertemporal dimension of an economic system and to understand aspects of equilibrium, growth, and trade cycle that are neglected by usual-Keynesian and Neoclassical-macroeconomic models. This concept and a very convenient graphical illustration have been introduced by Hayek (1935, 1941), and then developed mainly by Rothbard (1962), Skousen (1990), Reisman (1996) and Garrison (2001). Reisman has offered one of the most detailed expositions and most thoroughgoing utilizations of the concept of structure of production so far (1996, chaps. 15–17). The present paper seeks to formalize some aspects of his approach in the important case of a proportional goods-in-process structure. "Goods-in-process" means that only circulating capital is taken into account, and "proportional" means that the proportion between capital goods and originary factors is the same in all stages. In this context are presented, first a systematic method for the calculation and illustration of the structures, second a series of mathematical formulas that relate the main macroeconomic variables (consumption and investment spending, interest rate, aggregate income of originary factors), third a very simple and previously unnoticed formula for the average length of the structure [$\lambda = I(1+i)/C$], and fourth an application of the model to the complex issue of macroeconomic dynamics, i.e., an analysis of how, at each stage, the rate of return and investment change during the deformation of a structure.[1]

[1]I wish to thank an anonymous referee whose remarks have greatly helped me to improve this paper. I remain, of course, the only one responsible for its content.

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The Structure of Production in Static Equilibrium

Hypotheses: The Proportional Goods-in-Process Structure of Production

The institutional setting is a market economy, in which production is carried out through successive stages. Consumer goods are produced by combining the two kinds of factors of production, originary factors and capital goods. Originary factors are factors that have not been produced, i.e., labor and land (ground and natural resources). Capital goods are produced means of production, resulting from a combination, carried out at the previous stage of production, of originary factors and of capital goods. The capital goods of this previous stage are themselves produced with the help of originary factors and capital goods used at the preceding stage, and so on *ad infinitum*[2] (see Fig. 1). This structure of production may be conceived as the production process during its historical development that goes back uninterruptedly to the very first time a capital good was produced (Böhm-Bawerk 1959, p. 86). But this backward-looking interpretation of production is not satisfactory because human action is always forward-looking (Mises 1998, p. 491). The structure should rather be conceived as a "synchronized" structure in which the different stages of production occur simultaneously. In this case, Fig. 1 shows a kind of "snapshot" of the production process during a given period of time, during a year for instance (Hayek 1941, pp. 115–16; Skousen 1990, pp. 196–97).[3]

In static equilibrium, the same structure recurs year after year. There is no entrepreneurship and only two kinds of economic agents or functions exist, owners of originary factors of production and capitalists. Capitalists advance wages and rents to owners of originary factors, buy capital goods from the capitalists of the previous stage, combine these inputs in a production process, and wait until the end of this process to get their net income which is an interest on invested capital. The following simplifying hypotheses will be adopted all along.[4]

> (1) *Goods-in-process structure.* (a) Each stage of production lasts 1 year. (b) All the annual exchanges take place on the first of January. The services of the factors of production are entirely paid at the beginning of each stage (i.e., of each year) and the totality of the product is sold exactly 1 year later. Hypotheses (a) and (b) imply that the equilibrium rate of interest is unique and that the price of a produced good is $p_G = (1 + i)p_F$, where p_F is the price of its factors and i the annual equilibrium rate of interest. (c) There are no durable capital goods: they are entirely used up during a stage of production. Durable originary factors (land) are rented each year by their owners; they are not sold at their capitalized

[2]The structures that will be described below have an infinite number of stages. This hypothesis is obviously unrealistic since after a number of stages the values become negligible and have no meaning anymore in the realm of human planning and acting. But this hypothesis permits to resort to a mathematical apparatus whose usefulness should be clear at the end of the paper.

[3]In the "synchronized" structures analyzed in the present paper, each economic actor makes a yearly plan as a producer and as a consumer, hence all the actions and the structures are indeed forward-looking.

[4]Reisman uses all of these hypotheses in his macroeconomic theory, including the hypothesis (2) below of a proportional structure (see Reisman 1996, pp. 844–46).

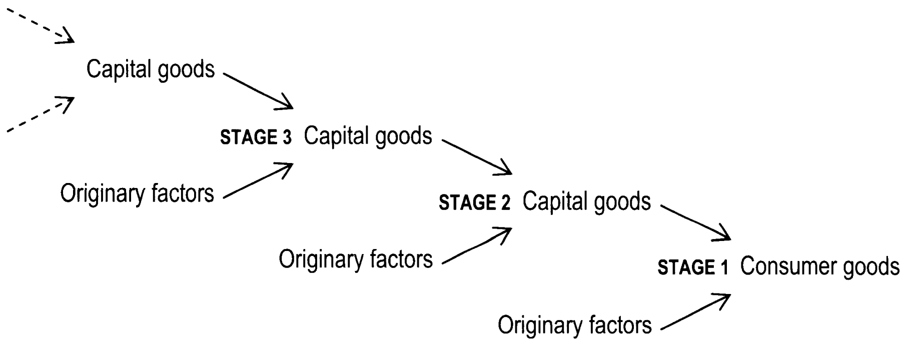

Fig. 1 The stages of a structure of production

value. This *goods-in-process* or "circulating capital" approach is typical of the Austrian perspective, contrary to the "fixed capital" approach (Hayek 1941, p. 47).

(2) *Proportional structure*. The proportion of the value of originary factors to investment is a constant across stages. Let n be a given stage of production. At this stage n, the total investment spending is I_n, and is the sum of expenditure I_{OFn} on originary factors and of expenditure I_{KGn} on capital goods: $I_n = I_{OFn}+I_{KGn}$. Hypothesis (2) means that the ratio $a_n = I_{OFn}/I_n$ is a constant: whatever the stage of production n, $a_n = a$. This is obviously the simplest hypothesis that can be made.[5] A structure exhibiting this feature—a constant proportion between originary factors and capital in all stages—will be called a *proportional* structure.

Determination of the Structure

In static equilibrium and under the hypotheses above, a structure is entirely determined by the three parameters C, i and a, where C is the annual aggregate spending on consumer goods, i the annual originary rate of interest, and a the ratio of originary factors to investment at each stage. The calculation begins at stage 1 and then proceeds backwards to stages 2, 3, 4, and so on (see Fig. 1).

Stage 1. The factors of production of stage 1 produce the consumer goods. Consumer goods have an annual aggregate value C. They are produced by the factors of production of stage 1, whose aggregate value is I_1 (investment at stage 1). In equilibrium, the sale of consumer goods just suffices to cover the expenses of production I_1 and the interest income iI_1 of the capitalists of stage 1: $C = I_1(1 + i)$. The value of I_1 is thus easily deduced from those of C and i.

[5] As Hayek (1941, p. 124) wrote: "It is perhaps reasonable to assume that the amount of input which is applied to the stock of intermediate products in each stage will bear a constant proportion to the amount of those intermediate products (or, in more popular but more inexact terminology, that the proportion between capital and labour will be roughly the same in all stages)."

Stage 2. The factors of stage 2 produce the capital goods of stage 1. Among the factors of production of stage 1, by definition only the capital goods have been produced. Their aggregate value is $I_{KG1} = (1 - a)I_1$ (since it follows from hypothesis (2) that the investment I_1 of stage 1 is the sum of an expense aI_1 on originary factors and an expense $(1 - a)I_1$ on capital goods). The capital goods of stage 1 are produced in the course of stage 2 by the factors of production of stage 2. In equilibrium, the sale of the capital goods of stage 1 I_{KG1} just suffices to cover the productive expenses I_2 of stage 2 and the interest income iI_2 on investment: $I_{KG1} = I_2(1 + i)$, or equivalently $(1 - a)I_1 = I_2(1 + i)$, and the value of I_2 is deduced from those of I_1, i and a.

Stage 3. The factors of stage 3 produce the capital goods of stage 2. The capital goods of stage 2 have an aggregate value $I_{KG2} = (1 - a)I_2$. They are produced by the factors of production used in stage 3. In equilibrium, the sale of the capital goods of stage 2 I_{KG2} covers the expenses I_3 of stage 3 and the interest income iI_3: $I_{KG2} = I_3(1 + i)$, or $(1 - a)I_2 = I_3(1 + i)$, and the value of I_3 is deduced from those of I_2, i and a.

Stage n ($n \geq 2$). The factors of stage n produce the capital goods of stage $(n - 1)$: $(1 - a)I_{n-1} = I_n(1 + i)$.

It is possible in this way to calculate step by step all the values that characterize the structure of production.

Calculation

Table 1 shows the data of the first ten stages of a structure characterized by a total annual consumption spending $C = 100$ monetary units (for instance: 1 unit = 10 billions $ or €), an equilibrium annual rate of interest $i = 10\%$, and a ratio $a = 20\%$ of originary factors at each stage.

At each stage n, aggregate investment I_n is calculated with the help of the algorithm of the previous Subsection (C and i determine I_1; I_1, i and a determine I_2; I_2, i and a determine I_3, etc.). The investments in originary factors and in capital goods are respectively deduced from the formulas $I_{OFn} = aI_n$ and $I_{KGn} = (1 - a)I_n$. The interest income Y_{in} on investment I_n is $Y_{in} = iI_n$. The values of Table 1 have been

Table 1 The first ten stages of the structure ($C = 100$, $i = 10\%$, $a = 20\%$)

Stage n	Investment I_n	Investment in originary factors $I_{OFn} = aI_n$	Investment in capital goods $I_{KGn} = (1 - a)I_n$	Interest $Y_{in} = iI_n$
1	90.91	18.18	72.73	9.09
2	66.12	13.22	52.89	6.61
3	48.08	9.61	38.47	4.81
4	34.97	6.99	27.98	3.50
5	25.43	5.09	20.35	2.54
6	18.50	3.70	14.80	1.85
7	13.45	2.69	10.76	1.34
8	9.78	1.96	7.83	0.98
9	7.11	1.42	5.69	0.71
10	5.17	1.03	4.14	0.52

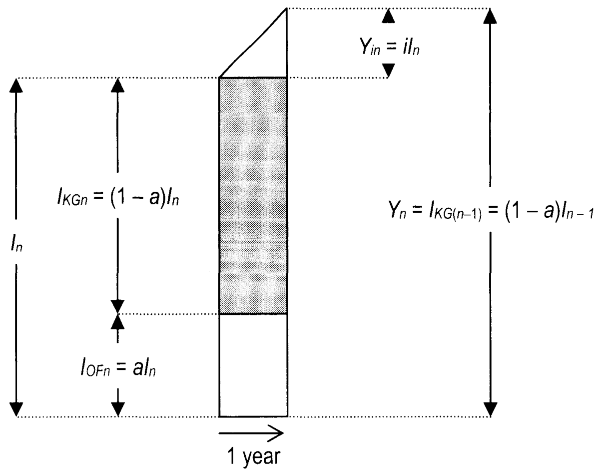

Fig. 2 A typical stage of production n

calculated with a computer program. Numbers are rounded at the second decimal, which explains why some equalities are not verified.

Three other important aggregate variables can be calculated:

- The total annual investment spending[6] $I = I_1 + I_2 + I_3 + \ldots + I_n + \ldots = \Sigma I_n$; for the structure ($C = 100$, $i = 10\%$, $a = 20\%$), this infinite sum converges[7] towards $I = 333.33$,
- The total annual income Y_{OF} of the owners of originary factors (aggregate wages and rents): $Y_{OF} = I_{OF1} + I_{OF2} + I_{OF3} + \ldots + I_{OFn} + \ldots = aI_1 + aI_2 + aI_3 + \ldots + aI_n + \ldots = a\Sigma I_n = aI = 0.2 \times 333.33 = 66.67$,
- And the total annual net income Y_i of the capitalists (aggregate interest): $Y_i = iI_1 + iI_2 + iI_3 + \ldots + iI_n + \ldots = i\sum I_n = iI = 0.1 \times 333.33 = 33.33$.

Illustration

A typical stage of production is represented in Fig. 2. Time runs from left to right. Nominal values are measured along the vertical axis. The vertical side on the left of the quadrangle is the investment spending I_n of stage n, split into an investment in originary factors I_{OFn} and an investment in capital goods I_{KGn}. The vertical side on the right is the gross income Y_n of the capitalists of stage n (it is equal to the value $I_{KG(n-1)}$ of the capital goods that are bought by the capitalists of stage $n - 1$). The net income of the capitalists of stage n is $Y_{in} = iI_n$ (interest on investment).

The whole structure ($C = 100$, $i = 10\%$, $a = 20\%$) is illustrated in Fig. 3. The vertical sides of the white rectangles show the value of the originary factors added at

[6] This investment comprises the expenditures made in order to buy all of the factors of production, including intermediate goods. It is of course much larger than the narrow "investment" used in the calculation of the GNP or GDP. See Rothbard (1962, pp. 340–41), Skousen (1991, Chapter 4) and Reisman (1996, Chapter 15).

[7] When more than 40 stages are taken into account, the calculation with the computer program of the sum $I_1 + I_2 + \ldots + I_n$ ($n \geq 41$) gives exactly the same result up to the third decimal (= 333.333).

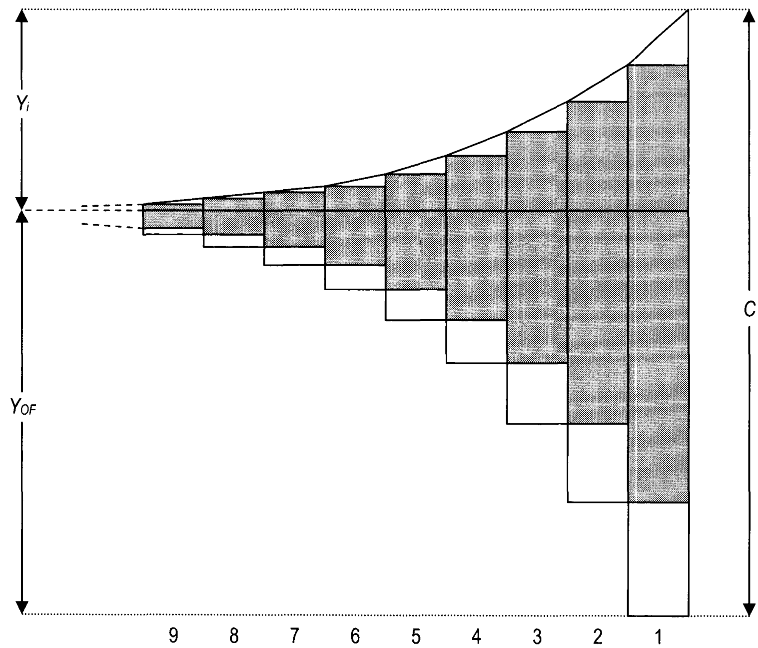

Fig. 3 An equilibrium proportional goods-in-process structure of production ($C = 100$, $i = 10\%$, $a = 20\%$)

each stage, and the horizontal sides show the duration of the process in which these factors are used (1 year). The grey rectangles show the value and period of production of the capital goods used at each stage. The vertical side of the white triangles show the interest accruing to capitalists at the end of each stage. The sum of the interest incomes through all stages is Y_i, the aggregate interest received by the capitalists in a year. The sum of the values of originary factors through all stages is the aggregate value Y_{OF} received by their owners in a year. The annual spending on consumption C is the sum of the two aggregate incomes Y_i and Y_{OF}. The distribution of the added value between the capitalists and the owners of originary factors appears clearly on the figure.

It has already been pointed out that this structure may be interpreted in two ways. First, as the temporal unfolding of the production process, successive stages belonging to successive years. Second, and much more importantly, as a depiction of the simultaneous or "synchronized" processes of production of all the stages of production during a given year.

Mathematical Description of the Equilibrium Structure

Austrian economists are usually a bit skeptical about the use of mathematics in economics, and rightly so. From an Austrian point of view, mathematical formalization has undoubtedly had very harmful consequences on the theory of price. The works of Mises (1998) and Hayek (1948) on prices, competition, and collectivist planning do not use any mathematical formula, and provide nevertheless a much deeper understanding of the economic process than most-if not all-of the mathematical models piled up in contemporary standard microeconomics textbooks. But when formalization can be used within a valid and pertinent theory, such as the macroeconomics of the structure of production, it must not be rejected because, as will be seen below, it can disclose important relations that would not be available through a purely qualitative analysis.

The Values at a Stage n

Annual investment I_n at any stage n may be expressed as a function of the three parameters of the structure (C, i, a) and of the number n. Since:

$$I_1 = \left(\frac{C}{1+i}\right)$$

$$I_2 = I_1\left(\frac{1-a}{1+i}\right) = \left(\frac{C}{1+i}\right)\left(\frac{1-a}{1+i}\right)$$

$$I_3 = I_2\left(\frac{1-a}{1+i}\right) = \left(\frac{C}{1+i}\right)\left(\frac{1-a}{1+i}\right)^2$$

it is obvious that:

$$I_n = \left(\frac{C}{1+i}\right)\left(\frac{1-a}{1+i}\right)^{n-1}$$

The values of annual investment in originary factors and in capital goods at stage n are then easily calculated:

$$I_{\mathrm{OF}n} = aI_n = \left(\frac{aC}{1+i}\right)\left(\frac{1-a}{1+i}\right)^{n-1}$$

$$I_{\mathrm{KG}n} = (1-a)I_n = C\left(\frac{1-a}{1+i}\right)^{n}$$

The gross income Y_n and net income Y_{in} of the capitalists of stage n are:

$$Y_n = I_n(1+i) = C\left(\frac{1-a}{1+i}\right)^{n-1}$$

$$Y_{in} = iI_n = \left(\frac{iC}{1+i}\right)\left(\frac{1-a}{1+i}\right)^{n-1}$$

The Aggregate Values

Total or aggregate investment I is the sum of the investments made throughout all the stages of production:

$$I = I_1 + I_2 + I_3 + \cdots + I_n + \cdots = \sum_{n=1}^{\infty} I_n$$

In order to calculate I, let us use the formula, given above, of I_n as a function of C, i, a and n:

$$I = \sum_{n=1}^{\infty} I_n = \frac{C}{1+i} \sum_{n=1}^{\infty} \left(\frac{1-a}{1+i} \right)^{n-1}$$

The infinite sum $(1 + q + q^2 + q^3 + \ldots + q^n + \ldots)$ converges[8] towards $1/(1 - q)$, if $0 \leq q < 1$. It suffices to define $q = (1 - a)/(1 + i)$ in order to reach the formula of total investment:

$$I = \frac{C}{1+i} \frac{1}{\left(1 - \frac{1-a}{1+i}\right)} = \frac{C}{i+a}$$

The final formula is surprisingly simple:

$$I = \frac{C}{i+a}$$

Investment spending I is directly proportional to consumption spending C and inversely proportional to the sum of the annual rate of interest i and the ratio a of originary factors at each stage.

The three other aggregate variables are the total annual interest Y_i, the aggregate income Y_{OF} of the owners of originary factors, and the aggregate income Y_{KG} of the owners of capital goods:

$$Y_i = \sum_{n=1}^{\infty} iI_n = i \sum_{n=1}^{\infty} I_n = iI = \frac{iC}{i+a}$$

$$Y_{\mathrm{OF}} = \sum_{n=1}^{\infty} I_{\mathrm{OF}n} = \sum_{n=1}^{\infty} aI_n = a \sum_{n=1}^{\infty} I_n = aI = \frac{aC}{i+a}$$

$$Y_{\mathrm{KG}} = \sum_{n=1}^{\infty} I_{\mathrm{KG}n} = \sum_{n=1}^{\infty} (1-a)I_n = (1-a) \sum_{n=1}^{\infty} I_n = (1-a)I = \frac{(1-a)C}{i+a}$$

Taking into account the relation $I = C/(i + a)$, it is perfectly equivalent to define a structure with parameters (C, i, a), or with (C, I, i), or with (I, i, a), or else with (C, I, a). Since $a = (C/I) - i$, aggregate incomes Y_i, Y_{OF} and Y_{KG} can for instance easily be expressed as functions of C, I and i:

$$Y_i = iI$$

$$Y_{\mathrm{OF}} = aI = \left(\frac{C}{I} - i\right)I = C - iI$$

$$Y_{\mathrm{KG}} = (1-a)I = \left[1 - \left(\frac{C}{I} - i\right)\right]I = I(1+i) - C$$

[8]Proof: define $s_n(q) = 1 + q + q^2 + \ldots + q^n$. Then $qs_n(q) = q + q^2 + q^3 + \ldots + q^{n+1}$, and $s_n(q) - qs_n(q) = 1 - q^{n+1}$. Therefore: $s_n(q) = (1 - q^{n+1})/(1 - q)$. If $0 \leq q < 1$ and if n becomes higher and higher, then q^{n+1} converges towards 0, and $s_n(q)$ thus converges towards $1/(1 - q)$.

Aggregate interest Y_i can also be written as a function of C, I and a:

$$Y_i = iI = \left(\frac{C}{I} - a\right)I = C - aI$$

Since aI is the final consumption of the owners of originary factors ($aI = Y_{\mathrm{OF}}$), and since in equilibrium the total final consumption is divided between these owners and the capitalists, then $(C - aI)$ is the final consumption of the capitalists, or what Reisman calls "net-consumption."[9] In equilibrium, aggregate interest is equal to net-consumption (Reisman 1996, Chapter 16).

The Average Period of Production

Since the number of stages is infinite, the *total* period of production is the sum of an infinite number of years and is thus also infinite. Böhm-Bawerk (1959 [1921], p. 86) defines the average period of production as the "*average* time interval occurring between each expenditure of originary productive forces and the final completion of the ultimate consumption good."[10] If each stage lasts 1 year, and if labor is homogeneous and is the only originary factor of production, then the average time interval between the expenditure of the quantity of labor L_1 used at the first stage and the production of the final consumption good is 1 year $\times$ (L_1/L), L being the total quantity of labor. The average time interval between the expenditure of the quantity of labor L_2 used at the second stage and the final consumption good is 2 years $\times$ (L_2/L), etc., the average time interval between L_n and the final consumption good is n years $\times$ (L_n/L), and so on. The sum of all these products is the Böhm-Bawerkian average period of production λ:

$$\lambda = \sum_{n=1}^{\infty} n\left(\frac{L_n}{L}\right)$$

Under the assumptions of the formal model, the originary factors used at stage n are measured by their aggregate value $I_{\mathrm{OF}n}$, and the total quantity of originary factors is measured by the total investment in originary factors $I_{\mathrm{OF}} = Y_{\mathrm{OF}}$:

$$\lambda = \sum_{n=1}^{\infty} n\left(\frac{I_{\mathrm{OF}n}}{Y_{\mathrm{OF}}}\right)$$

$I_{\mathrm{OF}n}$ and Y_{OF} have previously been expressed as functions of C, i, a and n:

$$\lambda = \frac{1}{Y_{\mathrm{OF}}}\sum_{n=1}^{\infty} nI_{\mathrm{OF}n} = \frac{i+a}{aC}\sum_{n=1}^{\infty} n\left(\frac{aC}{1+i}\right)\left(\frac{1-a}{1+i}\right)^{n-1} = \frac{i+a}{1+i}\sum_{n=1}^{\infty} n\left(\frac{1-a}{1+i}\right)^{n-1}$$

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[9] "Considered substantively, and essentially, net consumption is *the consumption expenditure of businessmen and capitalists*" (Reisman 1996, p. 725).

[10] For recent applications of the concept of period of production to production theory, see Reisman (1996) and Fillieule (2005).

Since, when $0 \leq q < 1$, the mathematical limit of the infinite sum $(1 + 2q + 3q^2 + \ldots + nq^{n-1} + \ldots)$ is[11] $1/(1-q)^2$, it suffices to replace q by $(1-a)/(1+i)$ in order to calculate the average period of production:

$$\lambda = \frac{i+a}{1+i}\frac{1}{\left(1-\frac{1-a}{1+i}\right)^2} = \frac{1+i}{i+a}$$

Since $i + a = C/I$, an equivalent formula, again surprisingly simple, is:

$$\lambda = \frac{I(1+i)}{C}$$

This formula is interesting in that it shows that a diminution of the rate of interest *by itself*—i.e., in the unrealistic case where i decreases without any change in the ratio (I/C)—would lead to a *shortening* of the structure.[12] In order for this kind of structure of production to become more roundabout, C must diminish or I must rise, or in other words people must spend less on consumption and more on investment. And if the interest rate diminishes, the magnitude of the lowering of C and of the rise in I must be important enough to more than offset the effect of this diminution. It may be added that a "feeble" relative variation of I and of C suffices to counterbalance an "important" relative variation of i. Consider a ratio $(I/C) = 2$, which fits with the data calculated by Skousen (1991, p. 45) for the economy of the United States. If the initial interest rate is 5%, then a 1% decrease in C and a 1% increase in I more than offset a 40% decrease in the interest rate (from $i = 5\%$ to $i' = 3\%$) and lead to a lengthening of the structure of production.[13]

Formal Macroeconomic Dynamics

A rigorous formalization of the structure is necessary in order to analyze accurately what happens when the shape of the structure changes, for instance in the standard Austrian case when a lowering of time preference leads to a decrease in C, a rise in I, and a decrease in i (Hayek 1935, pp. 44–54; Rothbard 1962, pp. 470–73; Skousen 1990, pp. 234–40; Garrison 2001, pp. 61–67). Our inquiry will here be restricted to the formal aspects of the problem, or in other words to the pure logic of structural change.

The Lengthening of the Structure

Let us consider the following process:

- Until year 2000, the structure of production (C, I, i) is in static equilibrium and remains the same year after year,

[11]If $s(q) = 1 + q + q^2 + q^3 + \ldots + q^n + \ldots = 1/(1-q)$, then its derivative is $s'(q) = 1 + 2q + 3q^2 + \ldots + nq^{n-1} + \ldots = 1/(1-q)^2$. (I wish to thank Alain Fillieule for this result.)

[12]It thus does not seem strictly correct in all cases to say that "When interest rates fall, the structure of production becomes more roundabout, redistributing marginal resources toward productive activities with lower rates of return" (Mulligan 2002, pp. 17–18).

[13]$\lambda = (I/C)(1+i) = 2(1+0.05) = 2.1000$ and $\lambda' = 2[(1+0.01)/(1-0.01)][1+0.05(1-0.40)] = 2.1016 > \lambda$.

- During year 2001 a lengthening—or capital deepening—of the structure occurs,
- This lengthening gives birth to a new structure (C', I', i') characterized by a lower consumption spending ($C' < C$), a higher investment spending ($I' > I$) and a lower rate of interest ($i' < i$),
- This new structure appears in 2001 and then remains the same during the following years.[14]

It is possible first to calculate the (*ex post*) aggregate interest in 2001, equal to the gross income of the capitalists in 2001 less the costs of production that they have incurred in 2000. Their gross income in 2001 is the income received in exchange for the sale of consumer goods C' and for the sale of capital goods $Y'_{KG} = I'_{KG}$. The costs of production are the productive expenditures made in 2000, and are equal to the investment spending I in year 2000. Aggregate interest in 2001 is thus[15]:

$$\widetilde{Y}_i = (C' + I'_{\mathrm{KG}}) - I$$

Since $I'_{\mathrm{KG}} = (1 - a')I'$ (see the Subsection on "The Aggregate Values"):

$$\widetilde{Y}_i = (C' - a'I') + (I' - I)$$

Aggregate interest is equal to the sum of the "net-consumption" of the *new* structure ($C' - a'I'$) plus the net investment ($I' - I$). As Reisman explains, when a net investment exists, "the amount of profit [called "aggregate interest" in the present paper] in the economic system turns out to equal the sum of net consumption plus net investment" (1996, p. 744).

With the help of the formal model, a second step can be taken in order to analyze the complex effects triggered on the rates of originary interest. The originary rate of interest of a stage n for year y is defined here as the *ex post* rate of return on investment. It is, in other words, the relative difference between the aggregate price of the products of this stage in year y and the aggregate price of the factors that have been used at the same stage n but bought during the *preceding* year ($y - 1$). In static equilibrium, under the assumptions of the model, there is one and only one originary rate of interest:

$$\forall n \geq 2, i_n = \frac{I_{\mathrm{KG}(n-1)} - I_n}{I_n} = \frac{(1-a)I_{n-1} - I_n}{I_n} = i$$

In year 2000 and before, the originary rate of interest is unique ($= i$). In year 2002 and after, the originary rate is unique again ($= i'$). But during the year 2001, the year when structure (C', I', i') replaces structure (C, I, i), *there are as many originary rates of interest as there are stages of production*. The originary rate of interest $\tilde{i}_n$ of a stage n ($n \geq 2$) is the relative difference between the income from the production of stage n in 2001 ($= I'_{\mathrm{KG}(n-1)} = (1 - a')I'_{n-1}$) and the costs of production, incurred in 2000, of the factors of stage n ($= I_n$):

$$\tilde{i}_n = \frac{(1 - a')I'_{n-1} - I_n}{I_n}$$

[14]It will of course take time until the new *equilibrium* structure is reached, since this equilibrium requires a reallocation of the convertible factors.

[15]The symbol "~" means that the value is calculated *between* two successive structures.

Table 2 Originary rates of interest between the structures ($C = 100$, $I = 300$, $i = 12\%$) and ($C' = 90$, $I' = 310$, $i' = 10\%$)

Stage n	Rate $\tilde{i}_n$ (%)
1	0.80
2	5.63
3	10.70
4	16.01
5	21.58
6	27.41
7	33.52
8	39.92
9	46.63
10	53.67

where I_n is the investment at stage n of the initial structure (C, I, i), I'_{n-1} the investment at stage $(n-1)$ of the new structure (C', I', i'), and a' the ratio of originary factors at each stage of the new structure. And for stage 1:

$$\tilde{i}_1 = \frac{C' - I_1}{I_1}$$

The rates $\tilde{i}_n$ are the sums of two components, the initial rate of interest i on the one hand, and a transitional rate of pure profit (or pure loss) on the other. They may be written accordingly:[16]

$$\tilde{i}_n = i + \frac{(1-a')I'_{n-1} - (1-a)I_{n-1}}{I_n} = i + \tilde{\pi}_n$$

In order to illustrate these formulas, let us consider the case of a lowering of the preference for the present that triggers an increase in I and a decrease in C and in i. The initial structure is for instance ($C = 100$, $I = 300$, $i = 12\%$) and the final structure ($C' = 90$, $I' = 310$, $i' = 10\%$).[17] The aggregate spending ($C + I$) remains constant[18] and the structure lengthens from $\lambda = (300/100)(1.12) = 3.36$ years to $\lambda' = (310/90)(1.10) = 3.79$ years. Table 2 presents the *ex post* originary rates of interest in year 2001, i.e. the rates between the two structures, for the first ten stages.[19]

The originary rate of interest is the higher, the further the stage is from final consumption. It is thus easy to explain why the convertible factors of production will tend to be reallocated towards the early—far from final consumption—stages: these factors have become relatively more profitable in the early than in the late stages. The two structures are displayed together in Fig. 4 (compare with the figures in

[16]The rate of pure profit or loss is the (momentary) difference between the actual rate of return of a producer and the equilibrium rate of interest (Rothbard 1962, p. 464). There is an ambiguity in the case analyzed here because the equilibrium rate itself is changing from i to i'. In the formula below the initial rate of interest i is used as the reference point, but it should be noted that this choice is in part arbitrary (the final rate i' could have been chosen instead).

[17]These data, specially the values of the rates of interest, have been chosen in order to facilitate the graphical representation.

[18]Reisman (1996) calls this assumption the "invariable money" hypothesis.

[19]In order to reduce the risk of error, two distinct computer programs have been used, one based on the step by step calculation of the Subsection "Determination of the Structure," and another based on the mathematical formula of I_n in the Subsection "The Values at Stage n." These two programs give the same results.

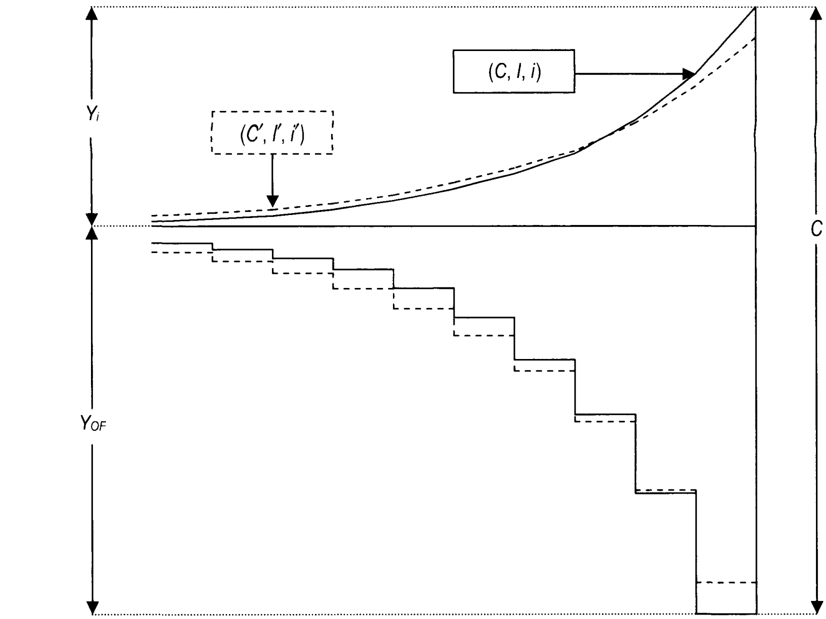

Fig. 4 A lengthening of the structure of production from ($C = 100$, $I = 300$, $i = 12\%$) to ($C' = 90$, $I' = 310$, $i' = 10\%$)

Hayek 1935, pp. 44 and 52; Rothbard 1962, p. 472; Skousen 1990, p. 235; Garrison 2001, p. 62).

Disentangling the Derived Demand Effect, the Investment Effect and the Discount Effect

The formal model offers another interesting possibility, which is to measure and to compare the respective effects on the structure of the variations in C, I and i. Garrison (2001, p. 64) describes the change of shape of a structure, in the case of a lengthening, by a sum of two effects:

(1) A "derived demand effect" explains the (vertical) narrowing of the late stages: the reduction in consumption spending implies a reduction of the (derived) demand of the factors that are used to produce consumer goods, and Garrison adds that "For stages of production sufficiently close to final output, this effect dominates";

(2) A "discount effect" explains the (vertical) widening of the early stages: "The reduction in the interest rate lessens the discount [at which labor is valued] and hence increases the value of labor. In the late stages of production, this effect is negligible; in the earliest stages of production, it dominates."

But what does it mean to say that an effect "dominates"? It means—this is our interpretation—that the consequences of this effect are greater than the consequences of other effects on the width of the structure. More precisely: at stage n, the effect of aggregate consumption C, for instance, "dominates" the effect of aggregate

Table 3 A comparison of the respective effects of C, I and I (structure $C = 100$, $I = 200$, $i = 5\%$)

Structure	Derived demand effect $C = 100 \rightarrow 99$ (-1%) [$I = 200 \rightarrow 200$] [$i = 5\% \rightarrow 5\%$]	Investment effect $I = 200 \rightarrow 202$ (+1%) [$C = 100 \rightarrow 100$] [$i = 5\% \rightarrow 5\%$]	Discount effect $i = 5\% \rightarrow 4.95\%$ (-1%) [$C = 100 \rightarrow 100$] [$I = 200 \rightarrow 200$]	Combined effects $C = 100 \rightarrow 99$ $I = 200 \rightarrow 202$ $i = 5\% \rightarrow 4.95\%$
Stage n	$e_{C,n}$ (%)	$e_{I,n}$ (%)	$e_{i,n}$ (%)	$\Delta I_n/I_n$ (%)
1	−1	0	0.05	−0.95
2	−0.10	0.90	0.00	0.79
3	0.81	1.81	−0.04	2.56
4	1.72	2.72	−0.08	4.36
5	2.65	3.65	−0.13	6.20
6	3.58	4.58	−0.17	8.06
7	4.52	5.52	−0.21	9.96
8	5.47	6.47	−0.25	11.90
9	6.43	7.43	−0.30	13.86
10	7.40	8.40	−0.34	15.86

investment I if the variation of I_n (investment at stage n) is greater when C changes than when I changes. In order to make this comparison, it is necessary to isolate the consequences of a relative change in C ($\Delta C/C$) from those of a relative change in I ($\Delta I/I$), and thus to calculate first the relative variation $\Delta I_n/I_n$ when C alone changes (I and i remaining constant), and second the relative variation $\Delta I_n/I_n$ when I alone changes (C and i remaining constant). If the relative variation $\Delta I_n/I_n$ is greater with a 1% change in C alone than with a 1% change in I alone, then we can say that the effect of C "dominates" the effect of I at stage n. Let us define the consumption-elasticity of investment at stage n as:

$$e_{C,n} = \frac{\left(\frac{I'_n - I_n}{I_n}\right)}{\left(\frac{C' - C}{C}\right)} = \frac{\left(\frac{\Delta I_n}{I_n}\right)}{\left(\frac{\Delta C}{C}\right)} \quad (I \text{ and } i \text{ being held constant})$$

This elasticity measures the effect of a relative variation of the annual consumption spending on the relative variation of investment at stage n, or in other words the impact of the variation of C—when I and i are kept constant—on the investment spending I_n at stage n. Similarly, an investment-elasticity and an interest rate-elasticity at stage n are defined as follows:

$$e_{I,n} = \frac{\left(\frac{I'_n - I_n}{I_n}\right)}{\left(\frac{I' - I}{I}\right)} = \frac{\left(\frac{\Delta I_n}{I_n}\right)}{\left(\frac{\Delta I}{I}\right)} \quad (C \text{ and } i \text{ being held constant})$$

$$e_{i,n} = \frac{\left(\frac{I'_n - I_n}{I_n}\right)}{\left(\frac{i' - i}{i}\right)} = \frac{\left(\frac{\Delta I_n}{I_n}\right)}{\left(\frac{\Delta i}{i}\right)} \quad (C \text{ and } I \text{ being held constant})$$

These elasticities are easy to calculate for a given structure and they show which effect dominates at a given stage of production. Table 3 gives the results in the case of the proportional goods-in-process structure ($C = 100$, $I = 200$, $i = 5\%$). The derived demand effect is calculated with a 1% decrease in C (from 100 to 99), the investment effect with a 1% rise in I (from 200 to 202), and the discount effect with a 1% decrease in i (from 5 to 4.95%).

These results corroborate Garrison's statement that the derived demand effect dominates the discount effect in the late stages.[20] It is however unexpected that the effect of the change in C on the relative variation of I_n does not diminish but *increases* when n goes up. Also unexpected is the fact that the effect of i is very small when compared to the effect of C. At stage 10, a 1% change in i induces a −0.34% relative change in I_{10}, while a 1% change in C induces a 7.40% relative change in I_{10}. Furthermore, it does not appear to be true, in the case of this proportional goods-in-process structure, that the discount effect dominates the derived demand effect in the earliest stages. In fact, the derived demand effect remains much higher than the discount effect for all the stages within reach of calculation by the program. Even when i is divided by 5 (from 5 to 1%), the effect of a 1% change in C dominates the discount effect in the early stages (at all stages, $|\Delta I_n / I_n|$ is higher with a 1% change in C than with a 80% change in i). As far as the investment effect is concerned, on the other hand, it is not a surprise to find that it dominates the two other effects when n rises (it is a bit surprising however to notice that it very quickly—as early as stage 2—exceeds the derived demand effect). An important remark must now be made. The separate study of the effects of C, I and i must not be interpreted as if these variables could change independently one from another. This is emphatically not the case: consumption, investment and the rate of interest are interrelated and *necessarily change together* when time preference changes in society. The analysis carried out above was aimed at showing the respective impacts of effects that happen all together on the structure.

Conclusion

This paper has expounded a formal model of the proportional goods-in-process structure of production and its two main applications, first the description of static equilibrium, and second the analysis of the dynamics or deformation of the structure. The contributions of the paper can be summed up in four points. (1) *Quantification.* Reisman (1996) has propounded a detailed method for calculating the values of this kind of structure. Here, his method is generalized to an algorithm that permits to calculate the spending/incomes at all the stages of a structure defined by three parameters: the annual consumption spending C, the annual interest rate i, and the ratio a of originary factors to investment at each stage (by definition of a proportional structure, a is the same in all stages). (2) *Illustration.* Most of the

[20] At stage 1 the derived demand effect is 1%, against 0.05% for the discount effect. At stage 2: 0.10% against 0.00%.

graphical representations of structures found in books and papers are very rudimentary. Only Rothbard (1962, p. 314) has attempted to apply the calculation of the stage-specific values to the illustration a structure. But his figure does not show the interest added at each stage. In other words, he has represented the "input curve" and not the "output curve" (in the sense of Hayek 1941). Yet, the output curve is the most interesting since it permits to visualize the accumulation of compound interest all along the production process. In the present paper, the graphical representations (Figs. 3 and 4) are based on accurate calculations. Furthermore, they show the distribution of added value between the owners of originary factors and the capitalists. (3) *Mathematical formalization.* Among the mathematical formulas demonstrated above, two stand out as specially noteworthy. First, the annual investment spending I is: $I = C/(i + a)$. The four parameters C, I, i, and a are thus mathematically related. As a consequence, a proportional goods-in-process structure in equilibrium is entirely determined by three parameters only chosen among C, I, i, a. Second: the average length of a structure is $\lambda = I(1 + i)/C$. The latter formula has not been noticed before, so far as the author can tell, and it shows that with this kind of structure the average length is directly—and not inversely—related to the rate of interest. (4) *Macroeconomic dynamics.* When the shape of the structure changes, the rates of return differ between stages. The formal model enables us to calculate these stage-specific rates, and to show that when a lengthening of a structure follows a lowering of time preference, the profitability is higher in the earlier than in the later stages; this helps us to explain in turn the reallocation of the convertible factors towards the early stages, and though this result is neither new nor unexpected, it can be here rigorously demonstrated.

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